$$= (x-\alpha)^{r-1}\left[(x-\alpha)\phi'(x) + r\phi(x)\right]$$

when $x = \alpha$;

$$(x-\alpha)\phi'(x) + r\phi(x) = r\phi(\alpha) \neq 0.$$

$\Rightarrow \alpha$ is an $(r-1)$ multiple root of

$$f'(x) = 0.$$

complete proof.

Remark:

If α is an $(r-1)$ multiple root of $f(x)=0$, similarly as previous theorem, we can see that α will be an $(r-2)$-multiple root of $f''(x)=0$; $(r-3)$-multiple root of $f'''(x)=0$, and so on.

Multiple roots:

If a root α of $f(x)=0$, repeats r times, then α is called an r-multiple root. A 2-multiple root is usually called a double root.

For example,

consider $f(x) = (x-2)^3 (x-5)^2 (x+1)$

Here 2 is a 3-multiple root, 5 is a double root and -1 is a single root of the equation $f(x)=0$.

Theorem: If α is an r-multiple root of $f(x)=0$, then α is an $(r-1)$ multiple root of $f'(x)=0$, where $f'(x)$ is derivative of $f(x)$.

Proof: Given that α is an r-multiple root of $f(x)=0$.

Then $f(x) = (x-\alpha)^r \phi(x)$ where $\phi(\alpha) \neq 0$.

Now, by applying product rule of differentiation, we obtain.

$$f'(x) = (x-\alpha)^r \phi'(x) + \phi(x)\, r \cdot (x-\alpha)^{r-1}$$

Also, from (1).

$$a_1 p^{n-1} q + \cdots + a_{n-1} p q^{n-1} + a_n q^n = -a_0 p^n$$

Dividing this expression by q, we get

$$a_1 p^{n-1} + \cdots + a_{n-1} p q^{n-2} + a_n q^{n-1} = \frac{-a_0 p^n}{q}$$

since the left side is an integer and q does not divide p, q most be a divisor of a_0. this complete the proof.

Corollary.

Every rational root of the equation $x^n + a_1 x^{n-1} + \cdots + a_n = 0$, where each a_i is an integer must be an integer. Moreover, every such root most be a divisor of the constant a_n.

proof:

This follows from the above (previous) theorem, by putting $a_0 = 1$.

Theorem: If the rational number p/q, a fraction in its lowest terms (so that p, q are integers prime to each other, $q \neq 0$) is a root of the equation $a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$ where $a_0, a_1, \cdots a_n$ are integer and $a_0 \neq 0$ the p is a divisor of a_n and q is a divisor of a_0.

Proof: since p/q is a root of given polynomial equation, we have

$$a_0\left(\frac{p}{q}\right)^n + a_1\left(\frac{p}{q}\right)^{n-1} + \cdots + a_{n-1}\left(\frac{p}{q}\right) + a_n = 0$$

multiplying by q^n, we get

$$a_0p^n + a_1p^{n-1}q + \cdots + a_{n-1}pq^{n-1} + a_nq^n = 0 \quad \text{--- (1)}$$

Dividing by p, we have

$$a_0p^{n-1} + a_1p^{n-2}q + \cdots + a_{n-1}q^{n-1} = \frac{-a_nq^n}{p}$$

now, the left side of the above equation is an integer and therefore $\frac{-a_nq^n}{p}$ is also must be an integer. since p and q have no common factor, p must be a divisor of a_n.

Page continued...

Theorem: In an equation with rational coefficients, the roots which are quadratic surds occur in conjugate pairs.

Proof: Let $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$, $a_0 \neq 0$, be an nth degree polynomial with rational coefficients.

Let $\alpha + \sqrt{\beta}$ is a root of $f(x) = 0$.

Divide $f(x)$ by $\left[\{x - (\alpha + \sqrt{\beta})\}\{x - (\alpha - \sqrt{\beta})\}\right]$
$= (x - \alpha)^2 - \beta$

Let $Q(x)$ be the quotient and $Ax + B$ be the remainder.

Proceeding exactly as in the previous question, we get $Ax + B = 0$

Thus we conclude that $\alpha - \sqrt{\beta}$ is also a root of $f(x) = 0$.