Differential equation

For B.sc / B.A Part II

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Equation reducible to linear equation.

An equation is of the form $\frac{dy}{dx} + Py = Qy^n$ where P & Q are function of x alone or constant. is called a Bernoulli's equation.

Such type of differential equation of the first order and first degree can be reduced into linear form as follows.

we have $y^{-n}\frac{dy}{dx} + Py^{-n+1} = Q$ (1)

Put $y^{-n+1} = v$, for reduce this equation into the linear form.

$\therefore (-n+1)y^{-n}\frac{dy}{dx} = \frac{dv}{dx}$ then (1) becomes

$$\frac{1}{1-n}\frac{dv}{dx} + Pv = Q.$$

or, $\frac{dv}{dx} + (1-n)P \cdot v = (1-n)Q$

which is a standard linear differential equation of the first order in v and it is solved by as usual.

Example : (i) $\frac{dy}{dx} - \frac{2y}{x} = y^4$

we have, $\frac{1}{y^4} \cdot \frac{dy}{dx} - \frac{2}{y^3} \cdot \frac{1}{x} = 1$ — (1)

$\left[\text{Put } \frac{1}{y^3} = z \Rightarrow -3\frac{1}{y^4}\frac{dy}{dx} = \frac{dz}{dx} \right]$

then (1) becomes

$$\frac{-1}{3} \cdot \frac{dz}{dx} - \frac{2z}{x} = 1$$

or, $\frac{dz}{dx} + \frac{6}{x} \cdot z = -3$ — (2)

which is a linear equation.

$$I \cdot F = e^{6\int \frac{dx}{x}} = e^{6\log x} = e^{\log x^6} = x^6 .$$

multiplying (1) by I.F and integrating, we get

$$z \cdot x^6 = -3\int x^6 dx \Rightarrow z \cdot x^6 = \frac{-3}{7} x^7 + c$$

which c is the constant of integration

or, $\frac{1}{y^3} \cdot x^6 = \frac{-3}{7} x^7 + c$

(ii) $xy^2\left(x\frac{dy}{dx} + y\right) = a^2$

sol^n: we have, $x^2y^2\frac{dy}{dx} + xy^3 = a^2$

Put $y^3 = z \Rightarrow 3y^2\frac{dy}{dx} = \frac{dz}{dx}$

$\Rightarrow y^2\frac{dy}{dx} + \frac{y^3}{x} = \frac{a^2}{x^2}$

or, $\frac{1}{3}\frac{dz}{dx} + \frac{z}{x} = \frac{a^2}{x^2}$

or, $\frac{dz}{dx} + \frac{3}{x}\cdot z = 3\cdot\frac{a^2}{x^2}$ —— (1)

which is linear equation in z.

$\therefore$ I.F $= e^{\int \frac{3}{x}dx} = e^{3\log x} = e^{\log x^3} = x^3$

multiplying (1) by I.F and integrating, we get

$$x^3\cdot z = 3a^2\int x\,dx$$

$$x^3y^3 = 3a^2\frac{x^2}{2} + K$$

where K is the constant of integration.

(iii) $x\frac{dy}{dx} + y = y^2\log x$.

solⁿ: Given equation $x\frac{dy}{dx} + y = y^2\log x$

or, $\frac{1}{y^2}\cdot\frac{dy}{dx} + \frac{1}{xy} = \frac{1}{x}\log x$

Put $\frac{1}{y} = z \Rightarrow \frac{-1}{y^2}\frac{dy}{dx} = \frac{dz}{dx}$

or, $-\frac{dz}{dx} + \frac{z}{x} = \frac{+1}{x}\log x$

or $\frac{dz}{dx} - \frac{z}{x} = \frac{-1}{x}\log x$ —— (1)

which is linear in z.

$\therefore$ I.F $= e^{-\int\frac{dx}{x}} = e^{-\log x} = e^{\log x^{-1}} = \frac{1}{x}$

multiplying (1) by I.F and integrating, we get,

$$\frac{1}{x}\cdot z = -\int \frac{1}{x^2}\log x \, dx$$

put $\log x = t \Rightarrow \frac{1}{x}dx = dt$ and $x = e^t$

$$= -\int t e^{-t} dt$$

$$= (t+1)e^{-t} + K = (1+\log x)\cdot\frac{1}{x} + K$$

or $$\frac{1}{xy} = (1+\log x)\frac{1}{x} + K.$$

where K is the constant of integration.

Some unsolved question which you have to try to solve,

(i) $dx - dy(1+xy^2)xy = 0$

(ii) $\frac{dy}{dx} + \frac{y}{x} = y^2$

(iii) $(1+x^2)\frac{dy}{dx} - xy = x^3y^3$

(iv) $\frac{dy}{dx} - xy = x^3y^2$

(v) $\frac{dy}{dx} + 2y\tan x = y^2$

(vi) $dx = x\sin y(1-x\cos y)\,dy$

(vii) $xy - \frac{dy}{dx} = y^3 e^{-x^2}$

(viii) $\frac{dy}{dx} + a = xe^{-y}$.